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Let us consider the collapse of an empty cylindrical hollow in a gas with equation of state $p = S\rho^\varkappa$. The motion of the gas outside the hollow is described by the equations

$$\frac{\partial(u + ac)}{\partial t} + (u + c)\frac{\partial(u + ac)}{\partial r} + \frac{uc}{r} = 0,$$
$$\frac{\partial(u - ac)}{\partial t} + (u - c)\frac{\partial(u - ac)}{\partial r} - \frac{uc}{r} = 0,$$

(1)

where u is the velocity of the gas, c is the velocity of sound, r is a space coordinate, t is the time, $a = 2/(\varkappa - 1)$, ($\varkappa$ being the adiabatic exponent), S is the entropy of the flow, and S is a constant. The free boundary of the gas is a cylinder of radius R(t). On the free boundary, the pressure p is equal to 0 and the velocity of the gas coincides with the velocity of the free boundary: u = dR/dt.

We assume the solution to be self-similar, that is, invariant under transformations of the similarity group G_k:

$$t \to \beta^k t, \qquad r \to \beta r, \qquad u \to \beta^{1-k} u, \qquad c \to \beta^{1-k} c,$$

where k is a parameter in the group, called the self-similarity exponent. To solve the problem posed, we need to ascertain for what values k^* of the parameter k does there exist a solution of the system (1) with the corresponding boundary conditions. In all the cases considered, the value of k^* was determined numerically. The asymptotic dependence of k on $\varkappa$ which we indicate by writing $K(\varkappa)$, is given for large values of $\varkappa$.

The present problem, its statement, and methods of solving it are analogous to the problem on the collapse of an empty spherical hollow, the solution of which is described in detail in [1].

Section 1

The solution of this problem is simplified by the fact that the invariance of the equations in the system (1) and of the boundary conditions under the similarity

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*Numbers in the margin represent pagination in the foreign text.

group G_k enables us to reduce the system (1) to a system of ordinary differential equations. Specifically, if we take for the independent variables t and $\xi = r^{-k}t$, and if we take for the unknown functions $a(\xi)$ and $b(\xi)$ which are connected with the functions u and c by the relations

$$\frac{r}{kt} a(\xi) = u + c, \qquad \frac{r}{kt} b(\xi) = u - c,$$

we obtain instead of (1) the system

$$4a\xi(a-1)(b-1)\frac{da}{d\xi} = (a-1)M + (a+1)N,$$

$$4a\xi(a-1)(b-1)\frac{db}{d\xi} = (a+1)M + (a-1)N, \qquad (2)$$

where

$$M(a, b) = N(b, a) = (1-a)\{[(1+a)b + (1-a)a](1-b/k) + (a^2 - b^2)/2k\}.$$

We shall assume that $t < 0$ prior to the instant of focusing, that $t = 0$ at the instant of focusing, and that $t > 0$ after the collapse.

Thr requirement that the functions u and c remain bounded as $t \to 0$ leads to the condition

$$a(0) = b(0) = 0. \qquad (3)$$

By virtue of the self-similarity, the free boundary is the line $\xi = \xi_1 = \text{const.}$ Then, from the fact that $u = dR/dt$ along the line $\xi = \xi_1$, it follows that

$$a(\xi_1) = b(\xi_1) = 1. \qquad (4)$$

The condition that the velocity of sound be nonnegative imposes a restriction on the choice of functions $a(\xi)$ and $b(\xi)$, namely,

$$a \leqslant b, \qquad t < 0; \qquad (5)$$

$$a \geqslant b, \qquad t > 0. \qquad (6)$$

Thus, the restriction (3)–(6) is a system of boundary conditions for (2).

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Instead of the system (2), let us look at the equivalent system

$$\frac{db}{da} = \frac{(\alpha+1)M+(\alpha-1)N}{(\alpha-1)M+(\alpha+1)N},$$ (7)

$$\frac{d\xi}{da} = \frac{4\xi\alpha(1-a)(1-b)}{(\alpha-1)M+(\alpha+1)N}.$$ (8)

Integration of the system (7)–(8) reduces to integration of equation (7) and to a quadrature.

To solve the problem posed, we need to know the value k^* of the parameter k at which Eq. (7) has an integral curve connecting the singular points $a = 0$, $b = 0$ and $a = 1$, $b = 1$ that lies in the half-plane $b \geqslant a$ of the phase plane of the equation. Therefore, we shall henceforth be primarily interested in Eq. (7). It has nine singular points, a fact that complicates considerably the search for the value of k* and the solution corresponding to it.

We have

0: $a = 0$, $b = 0$ is a dictritical node for all values of k;

M: $a = 1$, $b = 1$ is a saddle for all values of k;

K: $a = k$, $b = k$ is a node for all values of k;

N_1 and N_2: $a = (\alpha-1)(k-1) \pm \gamma[(\alpha-1)^2(k-1)^2 + 1 - 2(1+\alpha)(k-1)]$, $b = 1$;
N_3 and N_4: $a = 1$, $b = (\alpha-1)(k-1) + \gamma[(\alpha-1)^2(k-1)^2 + 1 - 2(1+\alpha)(k-1)]$;

$$N_5': a = \frac{\alpha+\gamma 2}{\alpha+2}k, \quad b = \frac{\alpha-\gamma 2}{\alpha+2}k; \qquad N_5'': a = \frac{\alpha-\gamma 2}{\alpha+2}k, \quad b = \frac{\alpha+\gamma 2}{\alpha+2}k.$$

Let us look at the singular points N_1 and N_2. These are real for all $k < k_3$, where $k_3 = 1/(1+\gamma\alpha)^2 + 1$. In all the calculations that we have made, $\varkappa$ varied between 5/3 and 3. In this interval, the nature of the singular points, N_1 and N_2 is as follows: for every singular point, there exist numbers k_1 and k_2 such that $k_1 < k_2 < k_3$ and, for $k < k_1$, the corresponding singular point is a focus; for $k_1 < k < k_2$, it is a node, and, for $k_2 < k < k_3$, it is a saddle. The numbers k_1 and k_2 depend, as was stated above, on the point. For the singular point N_1, we have

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$$k_2(x) = x / (x + 1 - 1/\gamma 2),$$
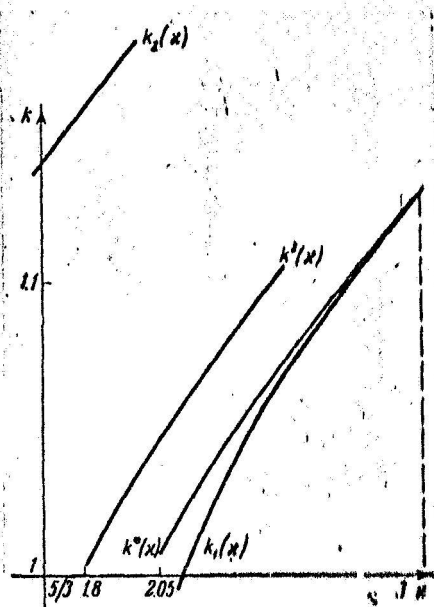
Figure 1 shows a graph of k_1 (x)

Figure 1.

The behavior of k_1 (x) was determined numerically. The relative position of the singular points and the isoclines of Eq. (7) are shown in Figs. 2 and 3. Figure 2 represents the case $k_1 < k < k_2$; Fig. 3 represents the case $k_2 < k < k_3$. Physical considerations, which are discussed in detail in [1], impose yet another restriction on the possible values of k, namely, the requirement that $k > 1$.

Let us turn now to the construction of the solution that we are seeking. Then, (1.1) is a saddle because this integral curve must be one of the separatrices of the singular point. The linearization (7) close to (1.1) shows that the saddle has two entrance directions: b-1 = a-1 and b-1 = 1-a. The first direction corresponds to the trivial solution a ≡ b; the second, to the solution that we are seeking. It follows from this that, for the solution to pass through the point (0, 0), it must intersect the line b = 1. (We note that the solution must lie in the region $b \geq a$, since in the present case $t < 0$.)

However, the trajectory can intersect the line b = 1 only at a singular point. If this were not the case, when we integrate equation (8) along the trajectory, the function $\xi(a)$ would be nonmonotonic, which is impossible.

It follows that the solution must intersect the line b = 1 either at the point N_1 or at the point N_2. Calculations have shown that in all cases the trajectory passes through N_1. The singular point N_1 corresponds to the characteristic of the system (1) that reaches the center at the instant of focusing of the hollow [1].

Therefore, for a solution to exist, it is necessary that N_1 be either a node or a saddle, that is, that $k_1 < k < k_3$. Also, k must be greater than 1. In general, for every value of k in the interval max $(1, k_1) < k < k_2$, the problem posed has a solution. The solution is not unique, but all solutions pass through the point N_1 with a weak discontinuity. In fact, the characteristic corresponding to N_1 was in no way singled out in the initial conditions. This leads to the natural requirement that the solution being sought be analytic at the point N_1.

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Numercial calculations of the analytic solution were made completely in analogy with the calculations for the collapse of a sphercial hollow.

The fundamental features of the method reduce to the following: two analytic /246 curves, the so-called separatrices, pass through the node N. It is just these separatrices that can serve as the solution of the problem posed. Therefore, to construct the solution that we are seeking, we need to find the value of the parameter k are which one of the separatrices passes through the points (0, 0) and (1, 1).

If the desired solution coincides with the separatrix of common direction, the numercial determination of k^* and the finding of the corresponding solution present considerable difficulties. In this case, the expansion of the solution close to N_1 is of the form

$$b - 1 = A_1(a - a_1) + A_2(a - a_1)^2 + \ldots + A_n(a - a_1)^n + C(a - a_1)^\lambda +$$
$$+ A_{n+1}(a - a_1)^{n+1} + \ldots,$$

where $(\alpha_1, 1)$ are the coordinates of N_1 and $\lambda(k, \varkappa)$ is the ratio of the eigenvalues of the matrix consisting of the linear terms in equation (7) obtained from the expansion of (7) close to the point N_1. Thus, from the point N_1 there issues a pencil of curves the first n terms in the expansion of all of which are the same in a neighborhood of the point N_1, and the problem consists in numerical determination of one of them, namely, the analytic curve corresponding in the expansion to the value C = 0. To single out this curve, we introduce the variables

$$\eta = \frac{b - 1 - A_1(a - a_1) - \ldots - A_n(a - a_1)^n}{(a - a_1)^{n+1}} - A_{n+1}, \quad x = a - a_1. \qquad (9)$$

At x = 0, the value of η is equal to 0 only for the analytic solution that we are /247 seeking. For the remaining curves in the pencil, the value of η at x = 0 is ∞.

Below, we shall describe the method of automation of the shift to the variables (9) proposed by Ya. M. Kazhdan. To do this, we make the change of variables $y = b-1$, $x = a-a_1$ in equation (7).

In the new variables, equation (7) becomes

$$\frac{dy}{dx} = \left(\sum_{l,m} a_{l,m} x^l y^m \right) \left(\sum_{l,m} b_{l,m} x^l y^m \right)^{-1}. \qquad (10)$$

The substitution (9) is done in n + 1 steps. The first is as follows:
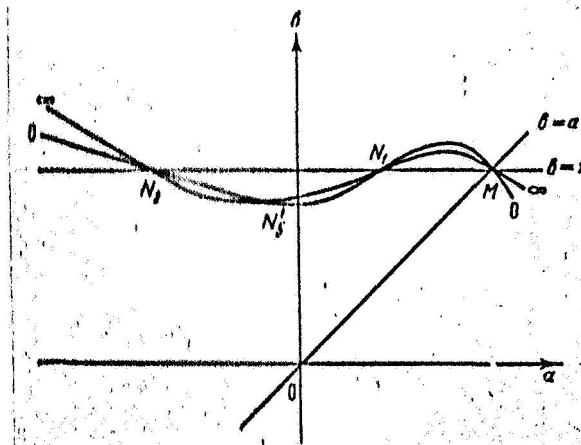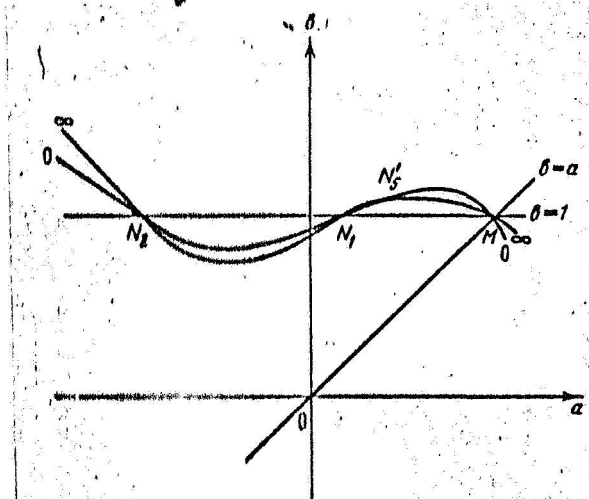
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Figure 2.

Figure 3.

$$y = x(y_1 + A).$$

Then,

$$\frac{dy_1}{dx} = \frac{1}{x}\left(\frac{dy}{dx} - A - y_1\right) = \frac{\Sigma a_{ih}' x^i y^h}{\Sigma b_{ih}' x^i y^h},$$

where the coefficients a_{ih}' and b_{ih}' are calculated from the formulas

$$b'_{n-2k,k} = \sum_{j=0}^{n-2k} b_{n-2k-j,k+j} \frac{(k+1)\ldots(k+j)}{j!} A^j, \quad n-2k \geq 0,$$

$$b'_{n-2k,k} = 0, \quad n-2k < 0,$$

$$a'_{n-2k,k} = a_{n-2k+1,k} - A b'_{n-2k+1,k} + b'_{n-2k,k+1}, \quad k \neq 0,$$

$$a'_{n,0} = a_{n+1,0} - A b'_{n+1,0},$$

where

$$a_{n-2k,k} = \sum_{j=0}^{n-2k} a_{n-2k-j,k+j} \frac{(k+1)\ldots(k+j)}{j!} A^j, \quad n-2k \geq 0,$$

$$a_{n-2k,k} = 0, \quad n-2k < 0.$$

In these substitutions, A assumes the values A_1, A_2, ..., A_{n+1} and is calculated as follows:

In the first step,

$$A_1 = \frac{a_{01} - b_{10} + \sqrt{[(a_{01} - b_{10})^2 + 4 b_{01} a_{10}]}}{2 b_{01}},$$

in all the remaining steps,

$$A_i = \frac{a_{10}}{b_{10} - a_{01}}.$$

Thus, the entire calculation is automated. Unfortunately, increase in the value of i is accompanied by a sharp increase in a_{ki}, b_{ki} and A_i, which causes considerable computational difficulties. In practice, we have succeeded in investigating on the machine only those values of k for which $1 < \lambda < 5$.

The calculations that we have made showed the following dependence of k* on ϰ:

ϰ = 2.05	k* = 1.00827	along a separatrix of common direction
ϰ = 2.10	k* = 1.0105	
ϰ = 2.15	k* = 1.0245	
ϰ = 2.20	k* = 1.0325	
ϰ = 2.35	k* = 1.0545	
ϰ = 2.50	k* = 1.07530	
ϰ = 3.00	k* = 1.13503	along an individual separatrix

The graph of $k^*(\varkappa)$ is shown in Fig. 1. One can see from that graph that the curve $k^*(\varkappa)$ must intersect the line $k = 1$ in a neighborhood of the point $\varkappa = 2$.

This is borne out by the calculations. Specifically, for $\varkappa = 2$ there is no index of self-similarity in the interval $1 < k < 1.035$. This brings up the thought that there may be either an index $k < 1$, or a second index of self-similarity. In Section 4, we shall show that there is no index of self-similarity for $k < 1$.

Section 3

After the focusing of the hollow as in the spherical case, a shock wave is reflected from the center. This is explained by the following considerations: We cannot assume the solution of our problem to be continuous. Therefore, the integral curve of equation (7) corresponding to the desired solution intersects the curve $a = 1$ at a point other than a singular point — as the calculations show. This leads to the fact that, when we integrate (8) along the integral curve $\xi(a)$, we obtain a nonmonotonic function, which, of course, is absurd.

Therefore, we assume the existence of a discontinuity in the solution the saltus at which is determined by the usual relations on a shock wave:

$$(D - u_1)^2 = V_1 \frac{p_2 - p_1}{V_1 - V_2}, \qquad (u_2 - u_1)^2 = (p_2 - p_1)(V_2 - V_1),$$

$$E_2 - E_1 = \frac{p_1 + p_2}{2}(V_1 - V_2). \tag{11}$$

Here, the subscript 1 corresponds to the state of the gas in front of the wave and 2 corresponds to the state behind the wave; D is the velocity of the shock wave; E is the internal energy; V is the specific volume.

If we introduce the notation $\eta = p_2/p_1$ and the self-similar functions $u = (r/t)U(\xi)$ and $c = (r/t)C(\xi)$, we can rewrite (11) in the form

$$\left(\frac{1}{k} - U_1\right)^2 = \frac{1}{\varkappa}\frac{\eta(\varkappa + 1) + \varkappa - 1}{2} \qquad \left(\frac{U_2 - U_1}{C_1}\right)^2 = \frac{2(\eta - 1)}{\varkappa[\varkappa - 1 + (\varkappa + 1)\eta]}$$

$$\left(\frac{C_2}{C_1}\right)^2 = \eta\frac{\varkappa + 1 + \eta(\varkappa - 1)}{\varkappa - 1 + \eta(\varkappa + 1)} \tag{12}$$

which is more convenient for determining the shock wave. To study the behavior of the solution after focusing for large values of ξ, we note that, at the center, the velocity u is equal to 0 and the velocity of sound is finite. Then, as $\xi \to \infty$, the quantity $C(\xi) \to \infty$ and $U(\xi)/C(\xi) \to 0$, and we have the asymptotic behavior

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$$U = A_0 + \frac{A_0 - (1+k)A_0^2 + \alpha k A_0^3}{A k C^2} + \ldots, \quad A_0 = \frac{\alpha(k-1)}{2k}. \tag{13}$$

The equation for U(C) is of the form

$$C\frac{dU}{dC} = \frac{-U + \alpha(1-k)C^2 + (1+k)U^2 + 2kC^2U - kU^3}{-\alpha + 2[\beta + \alpha + k(\alpha-1)]U + k\alpha C^2 - (1+\alpha)kU^2}. \tag{14}$$

Integration of the equation for large values of C with initial conditions (13) gives the behavior of our solution for large ξ. Piecing of the integral curve that we have obtained and connecting (0, 0) and (1, 1) is done in accordance with formulas (11) and it gives the complete solution of the problem posed.

Section 4

Consider the case in which the parameter k in the system (2) is less than unity. The singular points and the isoclines of equation (7) are shown in Fig. 4.
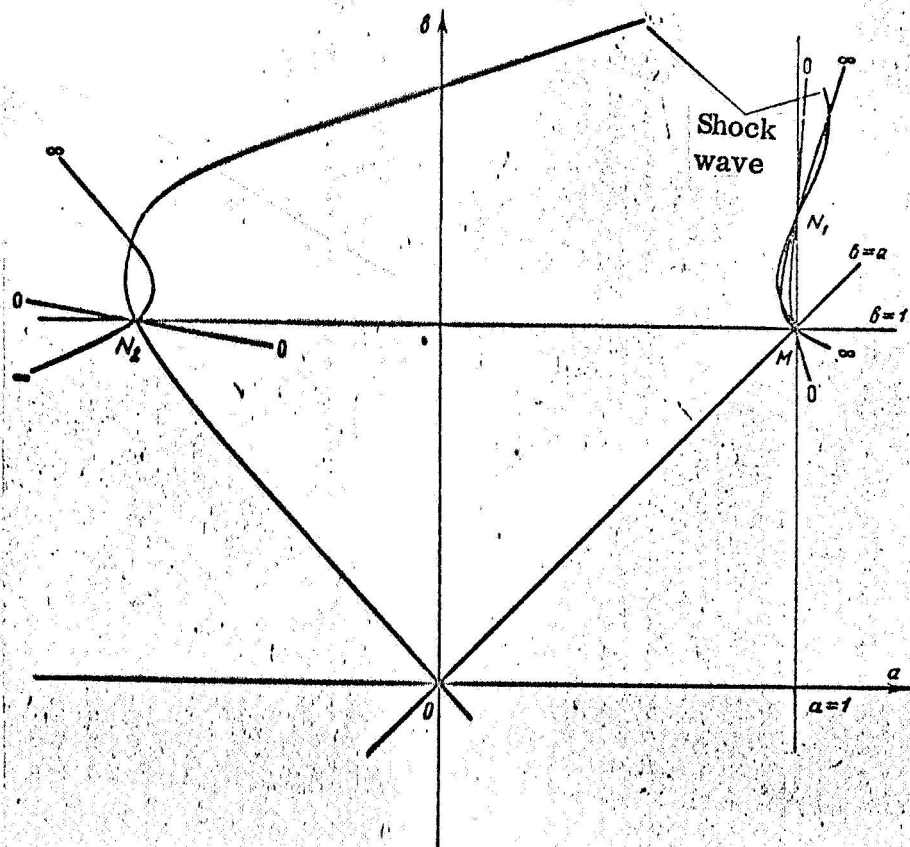

Figure 4.

From the locations of the isoclines close to the points M and N_1, we can easily see (see Fig. 5) that the solution issuing from the point M normally to the line b = a necessarily passes through the point N_1. This means that, for there to be a unique solution connecting 0 and M, we must restrict ourselves to the case in which N_1 is a node.
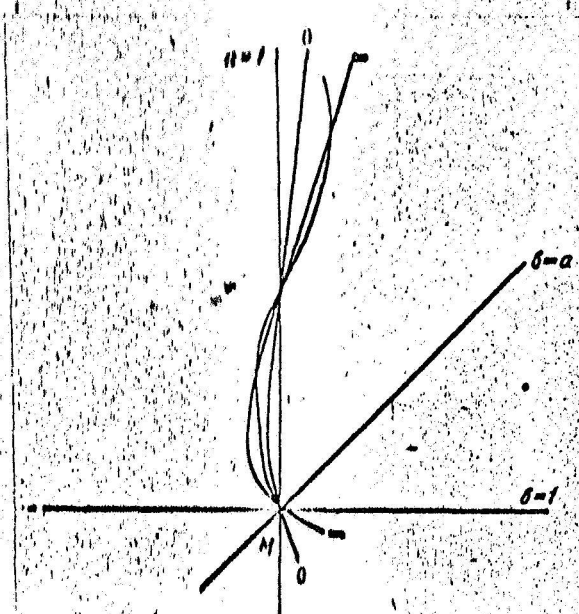

Figure 5.

It follows from this assumption that the solution passing through the node N_1 enters the region (a > 1, b > a). It is now obvious that there is no smooth solution connecting M and 0 since it must intersect the lines a = 1 and b = 1 at other than singular points, which would lead to nonmonotonicity of the functions $\xi(a)$ and $\xi(b)$.

Let us now suppose that there are discontinuities (shock waves) in the solution. Let us write the relationships that hold on a shock wave (11). In terms of self-similar variables, these are

$$a_2 = \frac{1}{2s(\varkappa+1)} \left((s+2)(\varkappa+1)s + 2w^2 - 2s^2 - ((2\varkappa s^2 - (\varkappa-1)w^2)((\varkappa-1)s^2 + 2w^2))^{1/2} \right)$$

$$b_2 = \frac{1}{2s(\varkappa+1)} \left((s+2)(\varkappa+1)s + 2w^2 - 2s^2 + ((2\varkappa s^2 - (\varkappa-1)w^2)((\varkappa-1)s^2 + 2w^2))^{1/2} \right), \qquad (15)$$

$$s = a_1 + b_1 - 2, \qquad w = a_1 - b_1.$$

The subscript 1 denotes the state of the gas in front of the wave, and the subscript 2 denotes the state behind it.

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In the ab-plane, the representation (15) has the following geometric interpretation (see Fig. 6): it maps the region $(a \geqslant 1, b \geqslant a)$ into the region

$$\{a \leqslant 1, b \geqslant 1 + ((\varkappa - 1 + 8\sqrt{[2(\varkappa - 1)\varkappa]})(a - 1) / (\varkappa + 1 - \sqrt{[2\varkappa(\varkappa - 1)]}\},$$

Here, the straight line a = 1 remains in position and the straight line b = a is mapped into

$$b = 1 + (\varkappa - 1 + 8\sqrt{[2\varkappa(\varkappa - 1)]}(b - 1)) / (\varkappa - 1 - \sqrt{[2\varkappa(\varkappa - 1)]}\}.$$

The region $(b \leqslant 1, b \geqslant a)$ is also mapped onto a triangular region adjacent to the line b = 1. The direction of the mapping is shown by arrows. We are interested only in the first region $(b \geqslant a, a \geqslant 1)$, since it is this region that the trajectory issuing from the point N_1 falls into. From this region, it makes a saltus into the region $(a \leqslant 1, b \geqslant 1)$, where a smooth solution can be constructed. However, there are a number of ways of choosing a weak discontinuity at the point N_1, constructing the shock wave, and hence gluing the smooth solution. If we choose the wave in such a way that the solution passes through the point N_2, we shall obtain one of the solutions of the boundary problem (see Fig. 4). However, the striking non-uniqueness of the choice of solution and the existence of weak discontinuities at the points N_1 and N_2 require supplementary restrictions.
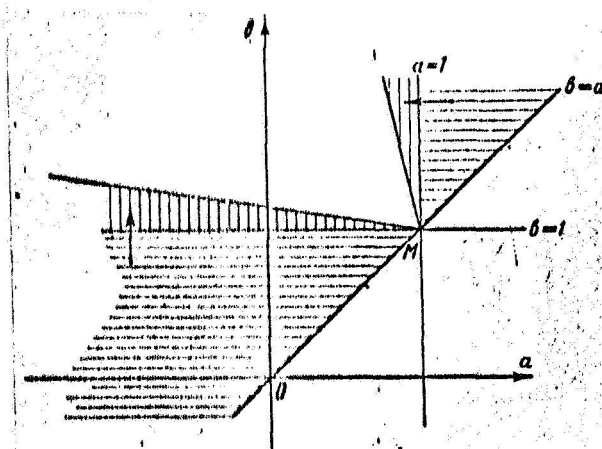

Figure 6.

By numerical integration, we have established that, at least in N_1, there are no analytic solutions in this family.

Thus, we have shown that there are no analytic indices of self-similarity for $k < 1$.

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In the case that we are considering, just as in the case of spherical symmetry, the analytic index of self-similarity is not a single-valued function of the adiabatic index $\varkappa$. Considerations on this point discussed in [1] give us a justification for assuming that corresponding to every value of $\varkappa$ is an even number of analytic indices of self-similarity.

Suppose that, for $\varkappa = 3$, there are an even number $k^1 < k^2 < \ldots$ of these indices.

Let us call the continuous function $k^n(\varkappa)$ that is the analytic index for all $\varkappa$ and satisfies the condition $k^n(3) = k^n$, where $n = 1, 2, \ldots$, the nth analytic index of self-similarity. Obviously, the function $k^*(\varkappa)$ that we have constructed is $k^1(\varkappa)$. As was indicated above, $k^*(\varkappa)$ ceases to exist when $\varkappa = 2$. Consequently, for $\varkappa \leqslant 2$ we get the smallest index $k^2(\varkappa)$. By numerical integration, we have found the following values of $k^2(\varkappa)$:

$\varkappa = 3.0$	$k^2(3) = 1.175$
$\varkappa = 2.5$	$k^2(2.5) = 1.105$
$\varkappa = 2.1$	$k^2(2.1) = 1.052$
$\varkappa = 2.0$	$k^2(2.0) = 1.036$
$\varkappa = 1.9$	$k^2(1.9) = 1.020$
$\varkappa = 1.8$	$k^2(1.8) = 1.0022$

The graph of $k^2(\varkappa)$ is shown in Fig. 1. One can easily see that $k^2(\varkappa)$ ceases to exist when $\varkappa \leqslant 1.8$. In an analogous way, we can conclude that, for every $n = 3, 4, \ldots$, there exists a $1 < \varkappa_n < 1.8$ at which the function $k^n(\varkappa)$, ceases to exist and there exists an interval $\Delta \varkappa_n \subseteq [1, 1.8]$, for which the nth of self-similarity exponent will be the smallest.

Section 5

As was pointed out in [1], p. 12, for the case of collapse of a spherical hollow, we can find the asymptotic behavior of $k^*(\varkappa)$ as $\varkappa \to \infty$. This asymptotic behavior corresponds to an incompressible liquid, which we should consider as the limiting case of a compressible liquid as $\varkappa \to \infty$.

We can derive analogously the asymptotic behavior in the cylindrical case. We note that (7) becomes degenerate as $\varkappa \to \infty$ (as $a \to 0$). Therefore, we make the substitution $z = \alpha(a - b)^2$ and then let α approach zero. When we do so, we get the equation

$$\frac{dz}{da} = \frac{2z}{3a} - \frac{8}{3}(k + a - 2),$$

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which we can solve. Its solution is

$$ z = Ca^{1/2} + 2a[4(2 - k) - a]. $$

The boundary conditions are as follows: as $a \to 0$, the point (0, 0) remains in position but the point (1, 1) is mapped into a = 1, z = 0.

The condition of analyticity of the solution leads to the result that C = 0. The condition that the solution passes through the point a = 1, z = 0 yields the result $k^*(\infty) = 7/4$.

In conclusion, the author expresses his deep gratitude to Ya. M. Kazhdan for his manuscript and to S. K. Godunov for his attention to the work.

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